On shape recognition

In honour of Pi Day

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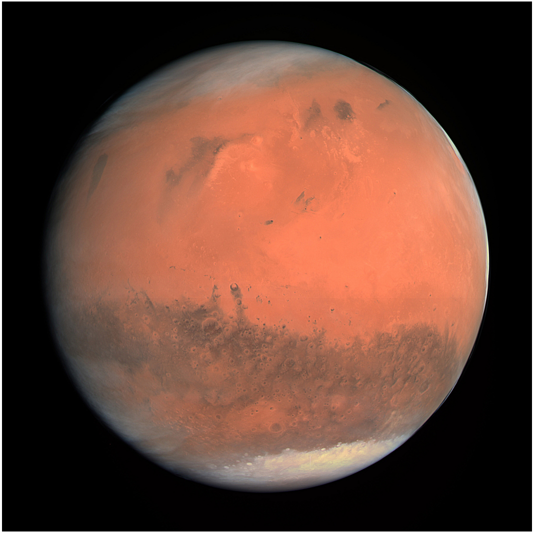
π.2021

Content

In the next 28 min., we are going to:

1. Look at colourful pictures.
2. See the origins and motivations for problems.
3. Solve 2 real-research problems: easy and not easy.
4. Stare at shapes to recognize them.
5. Homework: two open problems.

What do these shapes have in common?



Mars

What do these shapes have in common?

The Great Pyramid, 2570 B.C.

What do these shapes have in common?

A yummy grape.

What do these shapes have in common?

The British £1 coin (until 2017).

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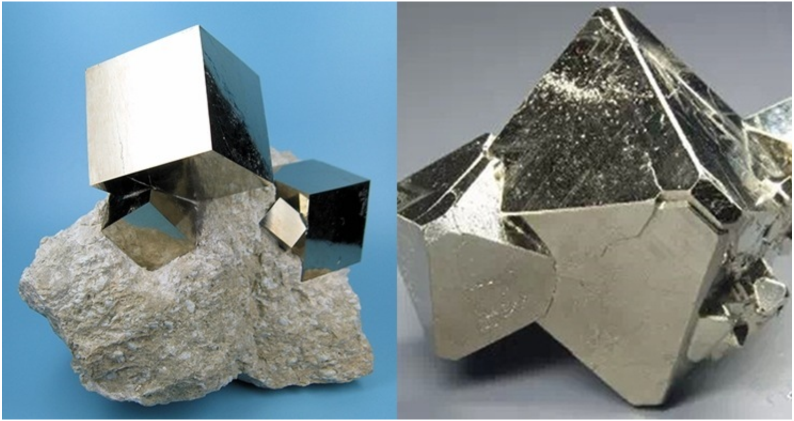


The mineral *pyrite*, also known as "**fool's gold**".

What do these shapes have in common?

Melting ice.

What do these shapes have in common?

Honeycomb.

What do these shapes have in common?

Batteries.

What do these shapes have in common?

A *tipi* (also *tepee* or *teepee*).

What do these shapes have in common?

Orange slices.

What do these shapes have in common?

Bridge components.

What do these shapes have in common?

A football.

What do these shapes have in common?

An egg.

Convexity

Shape K is called **convex** if for any two points $x, y \in K$, the segment $[xy]$ connecting them lies in K entirely.

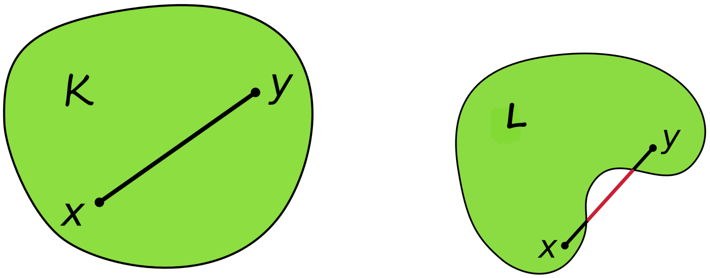
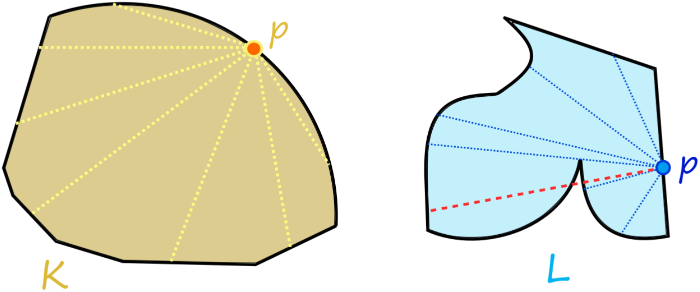


Figure: Convex shape K and non-convex shape L.

Every two points can see each other



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What does *Convex Geometry* study?

Convex Geometry is a branch of mathematics that studies compact convex shapes in Euclidean space which are called ***convex bodies***.

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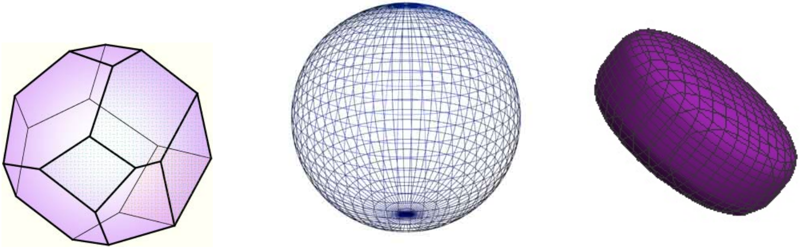

Figure: Examples: a polyhedron, a Euclidean ball, and who-knows-what.

How did it start?

Results and hints to problems which are of interest even today date back to antiquity 300-200 B.C.

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- *etc.*

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- **Algorithmic and complexity problems** *(computer science)*
- **Questions on min/max** *(theory of optimizations)*
- **Unique determination**
  (reconstruction of data, stability of shapes)

Question: uniqueness of rectangles

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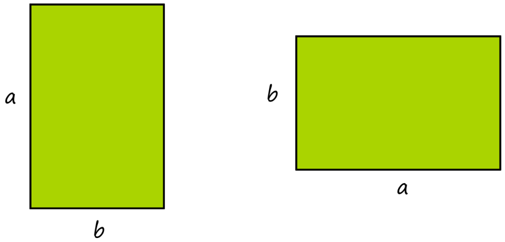

Figure: The same rectangle.

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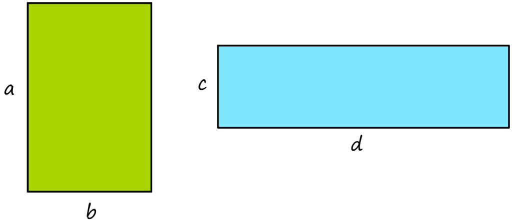

Figure: Different rectangles.

Area and perimeter

Let there be two different rectangles with the same A and P.

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$$(x-c)(x-d) = x^2 - \frac{P}{2}x + A = (x-a)(x-b).$$

This yields that $a = d, b = c$ **or** $a = c, b = d$.
So the rectangles must be the same.

Question: uniqueness of ellipses

Do the values of the *area* **and** the *circumference* determine an ellipse uniquely?

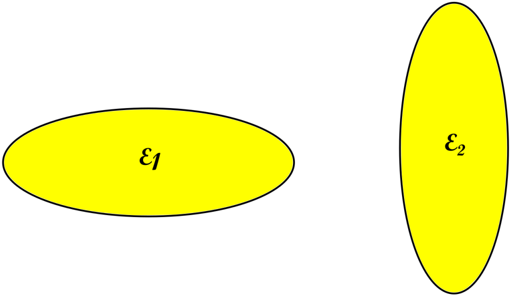

Figure: The same ellipse.

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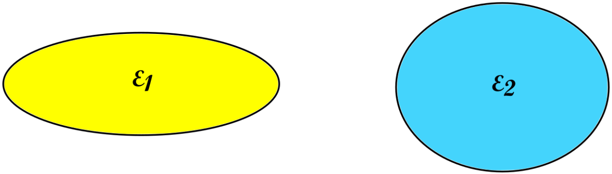
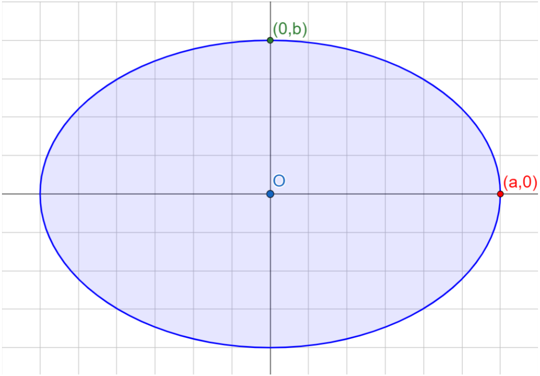


Figure: Different ellipses.

Who is an ellipse really?



$$\mathcal{E} = \left\{ (x, y) \in \mathbb{R}^2 : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

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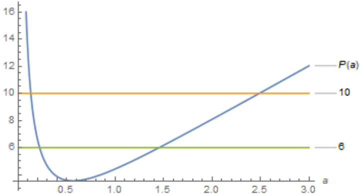
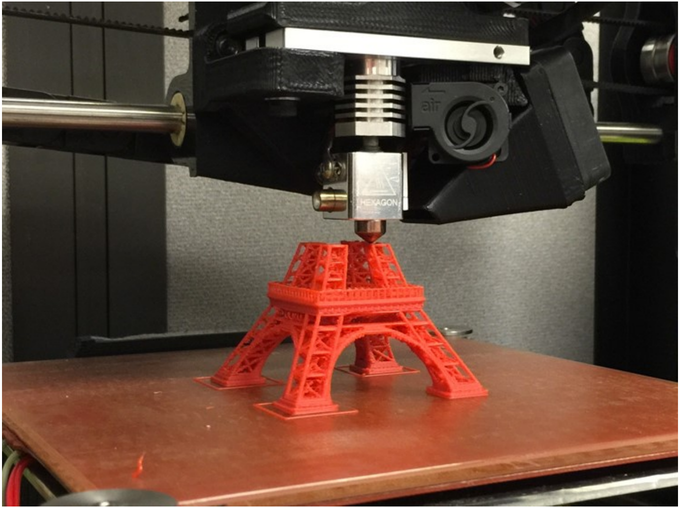
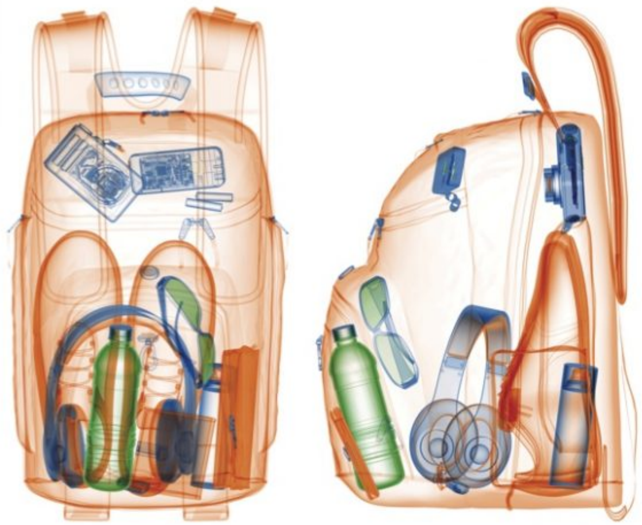


Equation $P(a) = const$ has **only two solutions** that correspond to the same ellipse with semi-axis a, b or b, a.

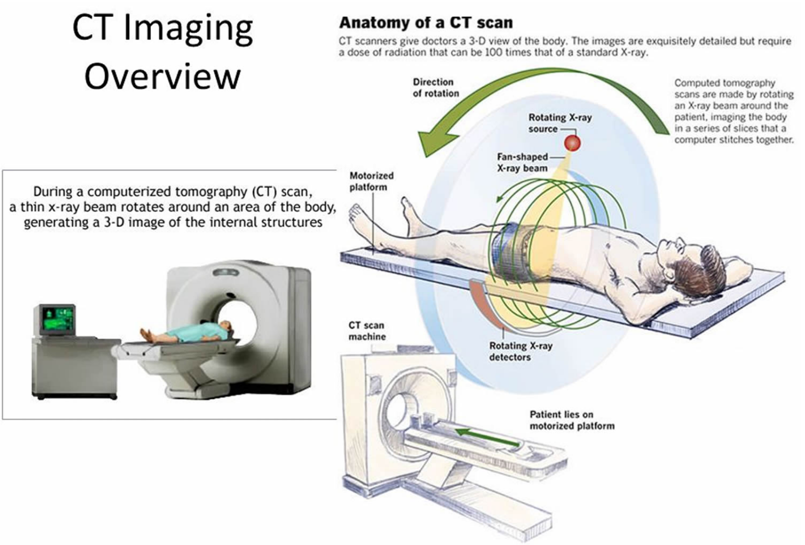
3D-printing



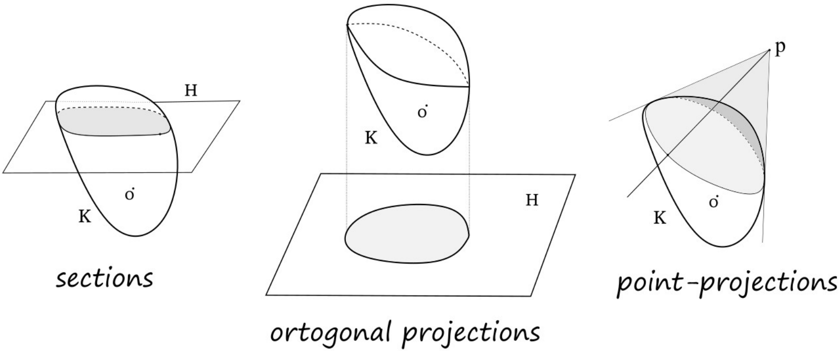
Airport X-ray



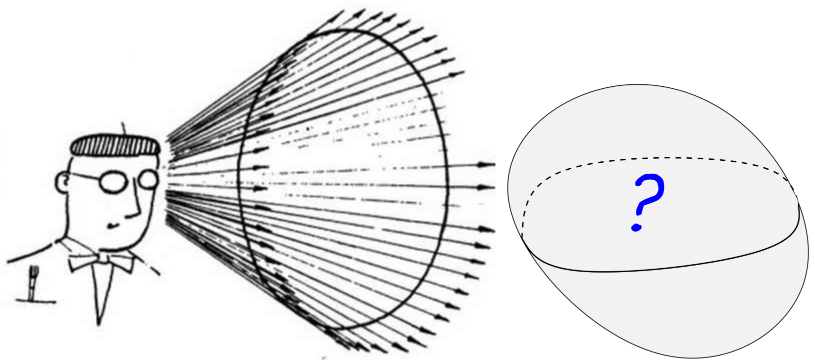
Computed Tomography Scan



Projections and sections

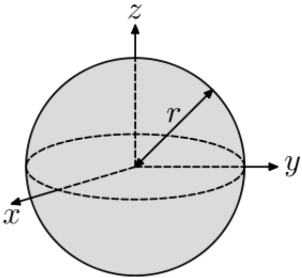


Sight recognition

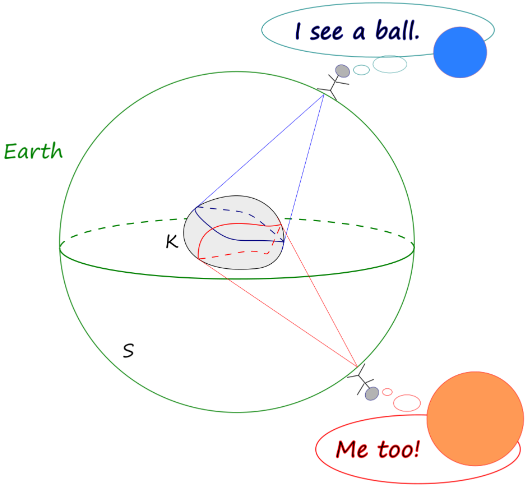


Who is a Euclidean ball?

$$\mathbf{B} = \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 \leq r^2\}.$$



A see-through planet: is K a ball?



Yes (Matsuura 1961).

Who are ellipsoids?

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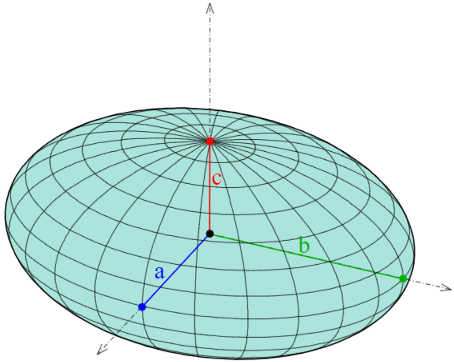
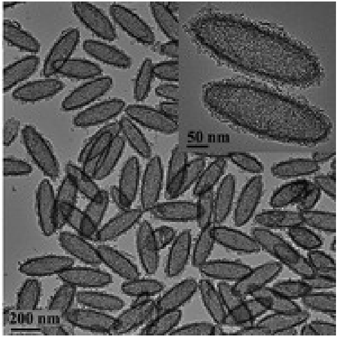
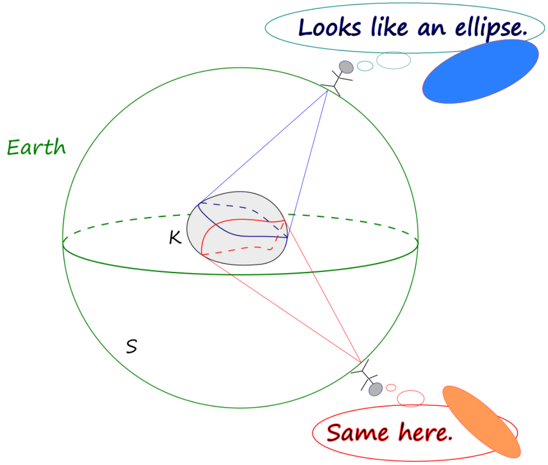


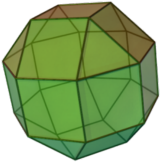
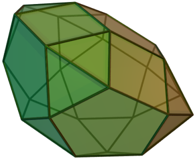
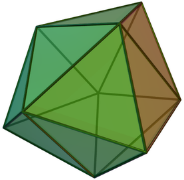


A see-through planet: is K an ellipsoid?

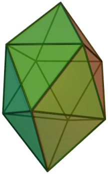
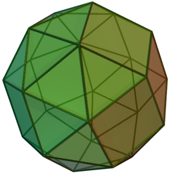
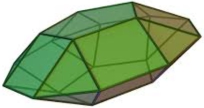


Yes (Bianchi, Gruber 1987).

Who are convex polyhedra?

The word **polyhedron** comes from the Classical Greek, as poly- ("many") + -hedron ("base" or "seat").

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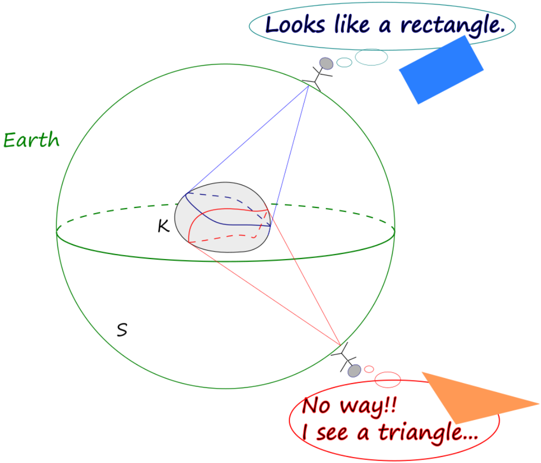
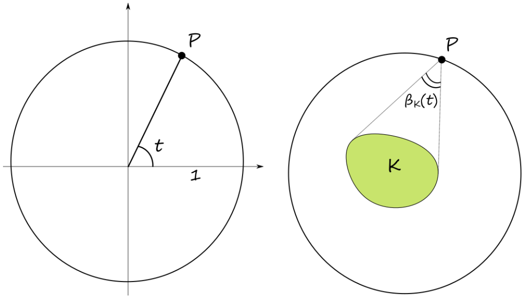


Yes (M. 2020).

Homework: Problem 1

Let K be a convex body in the unit disk, and define β_K to be the *visual angle* of K at P. This yields a function $y = \beta_K(t)$ on $[0, 2\pi]$.



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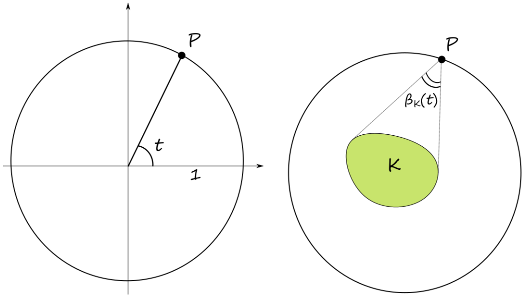

Q: Can there be another shape L with the same visual angles at every point P as K ($\beta_K(t) \neq const$)?

Homework: Problem 2

The diameter of a shape is the largest distance between any of its two points.

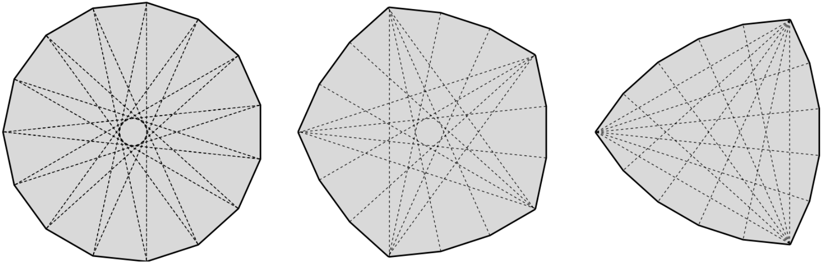

Figure: 15-gons of maximum perimeter with a unit diameter.

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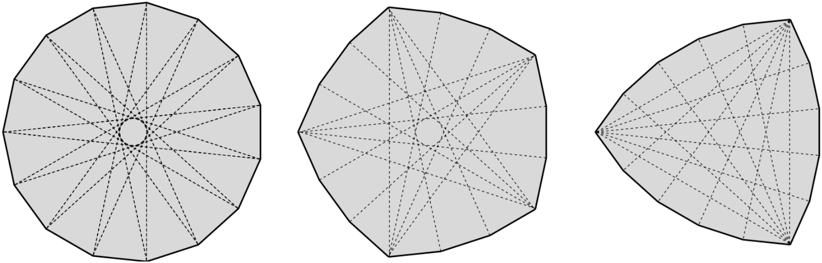

Figure: 15-gons of maximum perimeter with a unit diameter.

Q: What is the maximum perimeter of a convex 2^n-gon of unit diameter ($n \geq 3$)?

Thank You

Happy Pi Day