RESEARCH STATEMENT

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Contents

1. Introduction

My research interests lie mainly in Convex and Discrete Geometry with applications of Probability Theory and Harmonic Analysis to these areas. Convex geometry is a branch of mathematics that works with compact convex sets of non-empty interior in Euclidean spaces called convex bodies. The simple notion of convexity provides a very rich structure for the bodies that can lead to surprisingly simple and elegant results (for instance, [Ba]). A lot of progress in this area has been made by many leading mathematicians, and the outgoing work keeps providing fruitful and extremely interesting new results, problems, and applications. Moreover, convex geometry often deals with tasks that are closely related to data analysis, theory of optimizations, and computer science (some of which are mentioned below, also see [V]).

In particular, I have been interested in the questions of Geometric Tomography ([Ga]), which is the area of mathematics dealing with different ways of retrieval of information about geometric objects from data about their different types of projections, sections, or both. Many long-standing problems related to this topic are quite intuitive and easy to formulate, however, the answers in many cases remain unknown. Also, this field of study is of particular interest, since it has many curious applications that include X-ray procedures, computer vision, scanning tasks, 3D printing, Cryo-EM imaging processes etc.

I have also been fascinated by the broad field of geometric inequalities. This area is as old as geometry itself, the first results of which go back to Euclid's *The Elements*. Nevertheless, even nowadays it remains the center of a long list of many interesting mathematical topics. Several famous results (for instance, see [S]) include isoperimetric inequality, Brunn-Minkowski theorem, Blaschke-Santaló inequality, Grünbaum's inequality etc. Simultaneously, a lot of results can be generalized, and many questions remain open.

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Ongoing discussions with different specialists also made me curious about problems at the intersection of statistics and probability theory which is a fascinating and quite applicable area with numerous connections to data science and convex geometry. In particular, it is rich with tests related to comparison of data sets. Among those, computationally free or low-cost tests are the most beneficial. Not surprisingly, this has a direct connection to concentration inequalities ([BL]) and Kahane-Khintchine-type inequalities ([Ka]).

In what follows, I introduce necessary notions, open problems that have been considered, and state the results, as well as discuss ongoing projects and future plans.

2. Nakajima-Süss conjecture and its functional generalization

The beautiful world of geometric tomography flourishes with numerous problems on retrieval of information about sets from their projections, sections, or both. For example, in terms of data science, both of those procedures can be thought of as a restriction of data under certain conditions, i.e. *irrelevance criteria* or *information reduction procedure* (see [JL]). Hence, it is quite interesting to wonder how the restricted information can uniquely characterize the full data set. Examples of such conditions are volume estimates, rigidity of the structure or symmetry conditions. In this regard, see also *synchronization problems* and *questions of alignment* ([Ban]).

One of the curious applications of the above tasks is the *orientation estimation problem* which is of paramount importance and that appears in a Cryo-EM imaging process: a molecule is imaged after being frozen at a random (unknown) rotation and a tomographic 2-dimensional projection is captured. Given a number of tomographic projections taken at unknown rotations, we are interested in determining such rotations with the objective of reconstructing the molecule density.

From the point of view of convex geometry, this type of problems deals with *convex bodies* – compact sets K in a d-dimensional Euclidean space $\mathbb{E}^d$ with non-empty interior such that, for any two points $x, y \in K$ the interval connecting them, $tx + (1-t)y, t \in [0,1]$, belongs to K as well. Denote by $\mathbf{B}_2^d$ the unit Euclidean ball in $\mathbb{E}^d$, and by S^{d-1} the unit sphere. For any convex body $K \subset \mathbb{E}^d$ and a subspace $H \subset \mathbb{E}^d$ of dimension k, we can consider the orthogonal projection $K|H$ of K onto H. The following long-standing problem (cf., for example, [Ga, Problem 3.2, p. 125 and Problem 7.3, p. 289]) has inspired many related questions in my research.

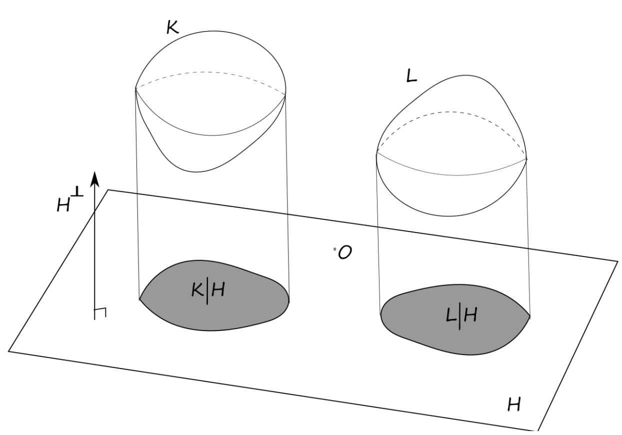


Figure 1. Projections coincide up to a congruency.

Problem 1. *Let $2 \leq k \leq d-1$. Assume that K and L are convex bodies in $\mathbb{E}^d$ such that the projections $K|H$ and $L|H$ are congruent for all subspaces $H \subset \mathbb{E}^d, \dim H = k$. Is K a translate of $\pm L$?*

Here we say that two sets $A, B \subset \mathbb{E}^k$ are *congruent (isometric)* if there exists an orthogonal transformation $\varphi \in O(k, H)$ in H such that $\varphi(A)$ is a translate of B.

A similar question can be asked about the sections of a convex body, namely

Problem 2. *Let $2 \leq k \leq d-1$. Assume K and L are star bodies in $\mathbb{E}^d$ such that the sections $K \cap H$ and $L \cap H$ are congruent for all subspaces $H \subset \mathbb{E}^d, \dim H = k$. Is $K = \pm L$?*

The first affirmative result for projections in the case of the trivial orthogonal transformations goes back to 1932 and belongs to S. Nakajima and W. Süss ([R3]). The first result for projections in the case of non-trivial orthogonal transformations was obtained by V. Golubyatnikov in the case $k = 2$ (he considered the case of directly congruent projections under an additional restriction on a lack of symmetries in the projections ([Go]). Some partial results were also proved by D. Ryabogin, M. Alfonseca, M. Cordier ([R2], [ACR]).

In our work with my PhD advisor we answered both questions in the class of convex polytopes. This was the first result where the case of opposite congruency was considered, i.e. orthogonal transformations including reflections.

Theorem 2.1 ([MR])**.** *Let $2 \leq k \leq d-1$ and let P and Q be two convex polytopes in $\mathbb{E}^d$ such that their projections $P|H$, $Q|H$, onto every k-dimensional subspace H, are congruent. Then there exists $b \in \mathbb{E}^d$ such that $P = Q + b$ or $P = -Q + b$.*

Theorem 2.2 ([MR])**.** *Let $2 \leq k \leq d-1$ and let P and Q be two convex polytopes in $\mathbb{E}^d$ containing the origin in their interior. Assume that their sections, $P \cap H$, $Q \cap H$, by every k-dimensional subspace H, are congruent. Then $P = Q$ or $P = -Q$.*

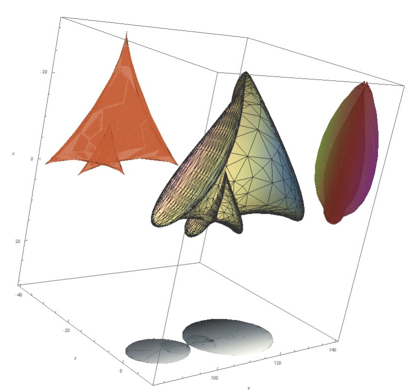


FIGURE 2. A classical hedgehog in $\mathbb{E}^3$ and a few of its projections.

According to the recent counterexamples in [Zh], one should suspect that convex bodies in the above problems must be congruent by more than a reflection. Although, almost nothing is known in the case when the convexity condition is dispensed with. In this regard, I extended the results of V. Golubyatnikov (see [Go]) to its functional counterparts. On the one hand, it is known that *a support function* h_K of a convex body K,

$$h_K(\xi) = \max_{x \in K} x \cdot \xi \quad \text{is continuous for all} \quad \xi \in S^{d-1}.$$

On the other hand, considering any twice-continuously differentiable function f on S^{d-1}, and treating it as a support function of a set (by considering the envelope of the hyperplanes with the equation $x \cdot u = f(u)$), we obtain the notion of a *classical hedgehog* H_f ([MM]).

These objects appear in differential geometry and can be highly non-convex. Following the ideas of V. Golubyatnikov and using the methods of harmonic analysis and algebraic topology, I have obtained the following functional generalization.

Theorem 2.3 ([M]). *Consider two classical hedgehogs* H_f *and* H_g *in* $\mathbb{E}^d, d \geq 3$. *Assume that their projections on any two-dimensional plane passing through the origin are directly congruent and have no direct rigid motion symmetries, then* $H_g = H_f + b$ *or* $H_g = -H_f + b$ *for some* $b \in \mathbb{E}^d$.

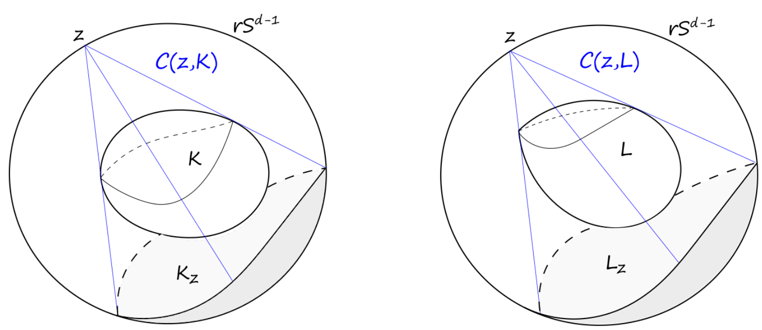
Quite often (for instance, see Section 5) problems of convex geometry can be re-stated in the analytic language of functions, which yields many fruitful generalizations and simplifies known techniques.

3. Visual recognition of convex bodies

Another approach to understanding the structures of convex bodies works with *point projections* and *visual cones*, which is exactly the way an observer "sees" an object, i.e. a data about a convex body is obtained. For example, this set of problems has a lot in common with *computer vision tasks*, [HZ].

The first result in this regard is a partial answer to the question posed by Kurusa ([KK]) regarding visually recognizing shapes of convex bodies from their shadows.

Problem 3. *If* K *and* L *are two convex bodies inside the sphere* rS^{d-1}, *and for each point* $z \in rS^{d-1}$ *the visual cones of* $C(z, K)$ *and* $C(z, L)$ *are congruent (the bodies look alike), then is it true that* $K = L$*?*



It was shown to be correct in the class of Euclidean balls ([Ma]). Besides, it was proved that if one of the bodies is an ellipsoid then so is the other one ([BG]). Also see [KO] for related discussions. I managed to provide the affirmative answer to the above problem for polyhedra, namely

Theorem 3.1 ([M1]). *Let* $d \geq 3, r > 0$ *and let* P, Q *be two convex polytopes contained in the interior of a ball* $r\mathbf{B}_2^d$. *Assume that for any point* z *on the sphere* $rS^{d-1} = \partial(r\mathbf{B}_2^d)$, *the support cones* $C(z, P)$ *and* $C(z, Q)$ *of the polytopes are congruent. Then* $P = Q$.

The proof of the above result suggests that the notion of *spherical projections* K_z and L_z that I introduced can be a useful technique. Interestingly enough, the following result holds (note that it is *not* equivalent to Theorem 3.1).

Theorem 3.2 ([M1]). *Let $d \geq 3, r > 0$ and let P, Q be two convex polytopes contained in the interior of a ball $r\mathbf{B}_2^d$. Assume that for any point z on the sphere $rS^{d-1} = \partial(r\mathbf{B}_2^d)$, the spherical projections P_z and Q_z of the polytopes are congruent. Then $P = Q$.*

Another most recent result of mine concerns two characterizations of polytopes in terms of non-central sections as well as point projections. Curiously enough, they imply their known counterparts for orthogonal projections and central sections ([Kl], [Z]).

Theorem 3.3 ([M2]). *Let K be a convex body in $\mathbb{E}^d, d \geq 3$, and $\{H_\alpha\}_{\alpha \in \mathcal{A}}$ be a set of k-dim planes, $2 \leq k \leq d-1$, all of which intersect the interior of K, such that:*

- *for any supporting line l of K, there exists a plane $H_\alpha \supset l$;*
- *for all $\alpha \in \mathcal{A}$, the intersection $K \cap H_\alpha$ is a k-dim polytope.*

Then K is a polytope.

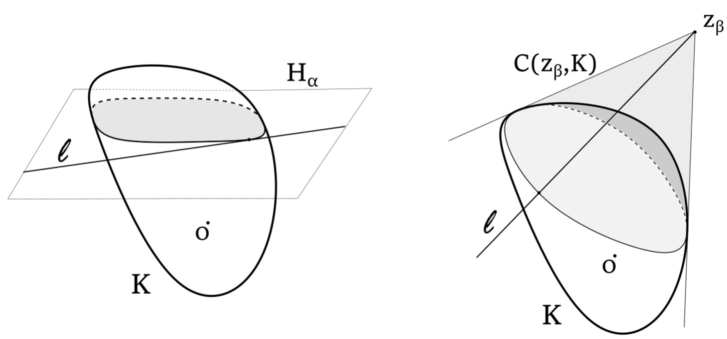


FIGURE 3. A $(d-1)$-dim section (left) and a d-dim visual cone (right) for a convex body K.

For instance, if δ is a continuous function on S^{d-1}, then set of hyperplanes $\{H_\xi\}_\xi$,

$$H_\xi = \{x \in \mathbb{E}^d : x \cdot \xi = \delta(\xi)\}, \quad \xi \in S^{d-1},$$

satisfies the conditions of the theorem ([BG]). In particular, for $\delta \equiv 0$, Theorem 3.3 implies the celebrated result of Victor Klee from 1959

Theorem 3.4 ([Kl]). *A bounded convex subset of $\mathbb{E}^d$ is a polytope if any of its k-dim central sections, $2 \leq k \leq d-1$, is a polytope.*

The version of Theorem 3.3 for ellipsoids was handled in [BG]. It also implies the corresponding result for Euclidean balls. Note that such settings are considered in several problems of Convex Geometry, such as questions related to characterizations of balls by sections and caps ([KO]), conical sections ([RY]), floating bodies ([Bl2], [BSW]), illumination bodies ([Sc], [W]), t-sections ([Y], [YZ]), and convex billiards ([G]).

Additionally, I proved the following dual result

Theorem 3.5 ([M2]). *Let K be a convex body in $\mathbb{E}^d, d \geq 3$, and $\{z_\beta\}_{\beta \in \mathcal{B}} \subset \mathbb{E}^d$ be a set of exterior points of K that satisfies:*

- *for any supporting line l of K, there exists a point $z_\beta \in l$;*
- *for a fixed k, $3 \leq k \leq d$, and all $\beta \in \mathcal{B}$, any k-dim visual cone $C_k(z_\beta, K)$ is polyhedral.*

Then K is a polytope.

For example, a closed surface S containing K in its interior satisfies the conditions of the point-set in Theorem 3.5. Also, when S is a convex surface, the analogous problem for circular cones was solved in [Ma]. In the class of elliptical cones, where $\{z_\beta\}_{\beta\in\mathcal{B}}$ is a closed set, the corresponding result was obtained for ellipsoids in [BG].

Lastly, by polar duality, Theorem 3.3 allows one to extend Theorem 3.5 to the case of infinitely distant points. This yields Klee's Theorem for orthogonal projections

Theorem 3.6 ([Kl])**.** *A bounded convex subset of $\mathbb{E}^d$ is a polytope if any of its k-dim orthogonal projections, $2 \le k \le d-1$, is a polytope.*

4. Intrinsic volumes of ellipsoids and the moment problem

The study of behavior of volume of a convex body under the Minkowski (vector) addition is the main focus of the Brunn-Minkowski theory. For a convex body K in $\mathbb{E}^d$, the classical Steiner formula asserts that for every $\epsilon > 0$,

$$\mathrm{vol}(K + \epsilon \mathbf{B}_2^d) = \sum_{i=0}^{d} \kappa_{n-i} V_i(K) \epsilon^{d-i},$$

where vol is the Lebesgue measure on $\mathbb{E}^d$, the addition $+$ is the Minkowski addition, and κ_{d-i} is the volume of $\mathbf{B}_2^{d-i}$. The coefficients $V_i(K)$ are known as the *intrinsic volumes* of K. The geometric interpretation of some of these quantities is the following: $V_d(K)$ is the *volume* of K, $V_{d-1}(K)$ is (a multiple of) the *surface area*, $V_1(K)$ is (a multiple of) the *mean width*. A thorough discussion of intrinsic volumes can be found in [S].

In 2017, Gusakova and Zaporozhets asked if an ellipsoid is uniquely determined (up to an isometry) by a tuples of its intrinsic volumes. Namely, they conjectured the following

Conjecture 4.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two ellipsoids in $\mathbb{E}^d$ such that $V_1(\mathcal{E}_1) = V_1(\mathcal{E}_2)$, $V_2(\mathcal{E}_1) = V_2(\mathcal{E}_2)$,..., $V_d(\mathcal{E}_1) = V_d(\mathcal{E}_2)$. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ are congruent.*

Petrov and Tarasov ([PT]) confirmed this conjecture in $\mathbb{E}^3$. For higher dimensions, the problem remains open.

My co-authors and I showed that a similar question for *dual volumes* has a positive answer in any dimension. Dual volumes were introduced by Lutwak ([L]) within the framework of the *dual Brunn-Minkowski theory.* In this theory the Minkowski addition of convex bodies is replaced by the radial addition of star bodies. The dual version of the Steiner formula asserts that

$$\mathrm{vol}(K \tilde{+} \epsilon \mathbf{B}_2^d) = \sum_{i=0}^{d} \binom{d}{i} \widetilde{V}_i(K) \epsilon^{d-i},$$

where K is a star body in $\mathbb{E}^d$ and $\tilde{+}$ is the radial addition. The coefficients $\widetilde{V}_i(K)$ are called the dual volumes. Note that $\widetilde{V}_d(K)$ is equal to the volume of K. Denoting by ρ_K the *radial function* of (a star body) K,

$$\rho_K(\xi) = \max\{a \ge 0 : a\xi \in K\}, \quad \xi \in S^{d-1},$$

one can write the dual volumes of K as follows:

$$\widetilde{V}_i(K) = \frac{1}{n} \int_{S^{d-1}} \rho_K^i(\theta)\, d\theta, \tag{1}$$

where the integration is with respect to the spherical Lebesgue measure.

Note that while the intrinsic volumes are invariant under translations, the dual volumes depend on the choice of the origin. Both the intrinsic volumes and dual volumes are invariant under orthogonal transformations.

The main result is the following

Theorem 4.2 ([MTY]). *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two ellipsoids in $\mathbb{E}^d$, $d \geq 2$, centered at the origin such that $\widetilde{V}_1(\mathcal{E}_1) = \widetilde{V}_1(\mathcal{E}_2)$, $\widetilde{V}_2(\mathcal{E}_1) = \widetilde{V}_2(\mathcal{E}_2)$,..., $\widetilde{V}_d(\mathcal{E}_1) = \widetilde{V}_d(\mathcal{E}_2)$. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ are congruent.*

As one can see, the right-hand side of formula (1) makes sense for all real i. This allows to use (1) as a definition of dual volumes of any order i. In view of this remark, in the statement of Theorem 4.2 the collection of the dual volumes $\{\widetilde{V}_i\}_{i=1}^d$can be replaced by any d-tuple of the form $\{\widetilde{V}_{i_k}\}_{k=1}^d$, where i_1,..., i_d are distinct non-zero real numbers from the interval $(-2, d]$. In some cases one can take numbers from a larger interval.

In the proof of Theorem 4.2 we showed that the question is in fact related to *a problem of moments* ([A]). Using this idea we also gave an alternative proof of the result of Petrov and Tarasov for the intrinsic volumes of ellipsoids in $\mathbb{E}^3$. Lastly, we managed to give partial affirmative answers to Conjecture 4.1 for ellipsoids of revolution.

5. Questions of geometric inequalities

The celebrated result of Grünbaum ([Gr], [Mi]) gives a lower bound for the volume of that portion of a convex body lying in a half space which slices the convex body through its *centroid (center of mass)*. More precisely, assume that the centroid of a convex body K is at the origin. Given a unit vector $\theta \in S^{d-1}$, we define a half-space $\theta^+ := \{x : \langle x, \theta \rangle \geq 0\}$. Then Grünbaum's inequality states that

$$\frac{\mathrm{vol}_d(K \cap \theta^+)}{\mathrm{vol}_d(K)} \geq \left(\frac{d}{d+1}\right)^d . \tag{2}$$

There is equality when, for example, K is the cone

$$\mathrm{conv}\left(\frac{-1}{d+1}\theta + \mathbf{B}_2^{d-1}, \frac{d}{d+1}\theta\right),$$

and $\mathbf{B}_2^{d-1}$ is the unit ball in $\theta^\perp$.

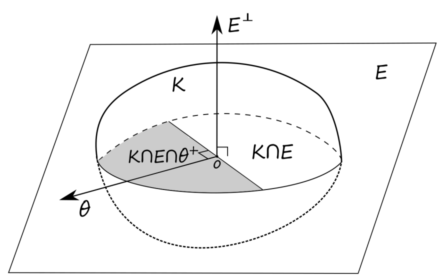


Figure 4. How small is $K \cap E \cap \theta^+$ compared to $K \cap E$?

Fradelizi, Meyer, and Yaskin in [FMY] asked the following question: what is the largest constant $c = c(d,k) > 0$, depending only on d and k, so that

$$\frac{\mathrm{vol}_k(K \cap E \cap \theta^+)}{\mathrm{vol}_k(K \cap E)} \geq c? \tag{3}$$

They showed that there is an absolute constant $c_0 > 0$ so that $c \geq c_1 := \frac{c_0}{(d-k+1)^2}\left(\frac{k}{d+1}\right)^{k-2}$, but they did not prove that it is optimal. Also, note that the value of c cannot be obtained from Grünbaum's inequality because the centroid of $K \cap E$ is in general different from the centroid of K.

The following result is a corollary of a more general statement regarding γ-concave functions that my collaborators and I proved. A particular case for log-concave functions ($\gamma = 0$) was shown in [MNRY].

Theorem 5.1 ([MSZ])**.** *Fix a k-dimensional subspace E of $\mathbb{E}^d$, and $\theta \in E \cap S^{d-1}$. Let $\widetilde{E}$ be the $(d-k+1)$-dimensional subspace spanned by θ and $E^\perp$. Let K be a convex body in $\mathbb{E}^d$ with the center of mass at the origin. Then*

$$\frac{\mathrm{vol}_k(K \cap E \cap \theta^+)}{\mathrm{vol}_k(K \cap E)} \geq \left(\frac{k}{d+1}\right)^k.$$

There is equality if and only if

$$K = conv\left(-\left(\frac{d-k+1}{k}\right)z + D_0,\ z + D_1\right),$$

where

- *$z \in E$ with $\langle z, \theta\rangle > 0$;*
- *D_0 is a $(k-1)$-dimensional convex body in $\widetilde{E}^\perp$;*
- *D_1 is an $(d-k)$-dimensional convex body in an $(d-k)$-dimensional subspace $F \subset \mathbb{E}^d$ for which $\mathbb{E}^d = span(E, F)$, and the centroid of D_1 is at the origin.*

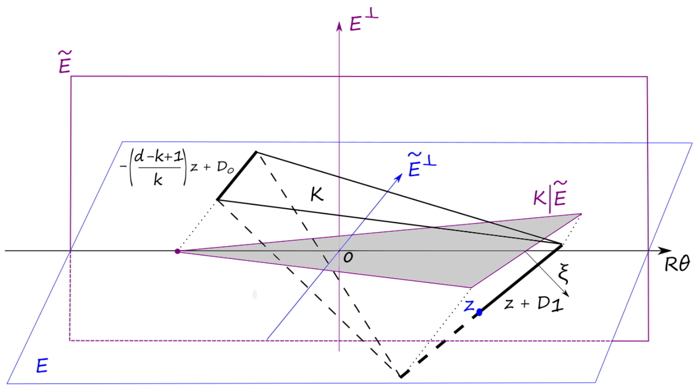


FIGURE 5. In $\mathbb{E}^3$ for $k = 1$, an optimizer is the convex hull of two segments in a general position.

Observe that the above result generalizes the one of Grünbaum (when $k = d$), and also provides the full characterization of the equality case.

6. Analytic permutation testing

The permutation (randomization) test is a versatile type of exact non-parametric significance test that requires drastically fewer assumptions than similar parametric tests, but achieves the same statistical power. The main downfall of the permutation test is the high computational cost of running it making such an approach laborious for complex data and experimental designs and completely infeasible in any application requiring speedy results.

My co-authors and I proposed a rectification of this problem through an application of Kahane-Khintchine-type inequalities under a weak dependence condition ([Sp]), and thus came up with *a computation free permutation test* which is valid for finite samples. The general framework can be applied to multivariate and functional data as well as the corresponding covariance matrices and operators resulting from theorems in commutative and non-commutative Banach spaces. We also showed that such an approach can be extended from two-sample to N-sample testing. This generalizes the approach for Rademacher sums to *Rademacher chaoses* ([V]).

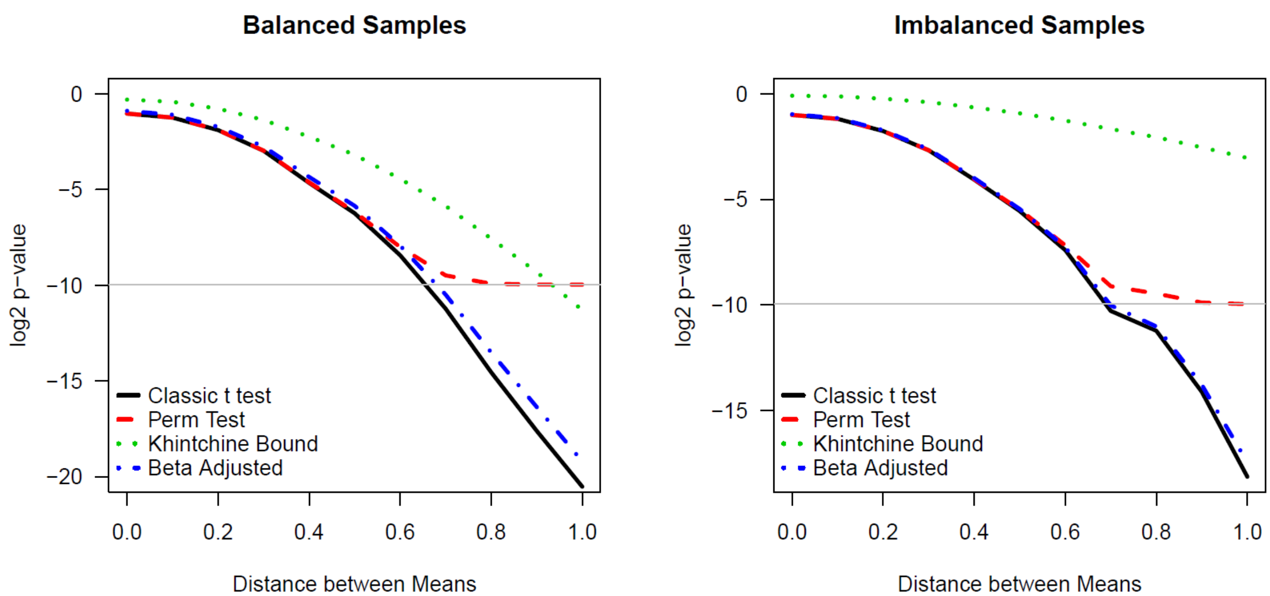


Figure 6. A comparison of univariate two sample tests for normal data with balanced sample sizes $m_1 = m_2 = 100$ ($\kappa = 1$), and for imbalanced $m_1 = 140; m_2 = 60$ ($\kappa = 2\frac{1}{3}$).

In particular, let $n = m_1 + m_2$, and $X_1, \ldots, X_n \in \mathbb{R}$ be independent random variables such that their mean $EX_i = \mu_1$ for $i \leq m_1$, and $EX_i = \mu_2$ for $i \geq m_1 + 1$. We wish to test the hypothesis $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. For this, we treat $X_1, \ldots, X_n \in \mathbb{R}$ as fixed and consider $\pi \in \mathbb{S}_n$ a random permutation uniformly distributed on the symmetric group of n elements. Then we consider the randomly permuted test statistic

$$T(\pi) = \frac{1}{s}\left[\frac{1}{m_1}\sum_{i=1}^{m_1} X_{\pi(i)} - \frac{1}{m_2}\sum_{i=m_1+1}^{n} X_{\pi(i)}\right],$$

which is normalized by the sample standard deviation s of the entire set $X_1, \ldots, X_n$. Let T_0 be the test statistic $T(\pi)$ for the original ordering (data set). Then, the p-value for the above hypothesis test is $P(T(\pi) \geq T_0)$, which is often approximated by randomly generating $N \ll n!$ random permutations from $\mathbb{S}_n$. To avoid the simulation-based approximation, we instead prove a sub-Gaussian *concentration inequality* on the p-value

Theorem 6.1 ([KMS])**.** *Let $m_1 = \kappa m_2$ for some $\kappa \geq 1$, then*

$$\mathbb{P}(T(\pi) \geq t) \leq \exp\left(- \frac{nt^2}{2[\kappa + 1]^3} \right).$$

We also provided extensions of our tail bound to commutative and non-commutative Banach spaces, and presented refined bounds using regularized incomplete beta function and Talagrand's concentration inequality ([T]). Lastly, as mentioned earlier, the generalization for N-sample testing, $N \geq 3$, was also shown.

To conclude, we successfully tested this methodology on classic functional data sets including the *Berkeley growth curves* and the *phoneme dataset*, and compared the outcomes with known approaches. We also considered hypothesis testing on speech samples, functional and operator data, under two experimental designs: *the Latin square* and the *complete randomized block design.*

7. Current work and future plans

I continue my investigations in the area of Convex Geometry and related to it topics of Harmonic Analysis and Probability Theory. Despite the fact that convexity is a very old topic that could be traced at least to Archimedes, it still offers a long list of very interesting open problems that are often accessible even to undergraduate students.

I am especially interested in the questions that lie at the intersection of Geometric Tomography and Computer Vision: unique determination of convex bodies under certain conditions on their different types of sections, projections, shadows or other quantitative characteristics that appear in **Brunn–Minkowski theory**. For example, questions involving brightness or width functions often are dealt with in pure differential geometry settings: investigation of solutions for non-ordinary differential equations on principal curvatures of a sufficiently smooth convex body (for example, see [HH]).

I am also trying my hands on some problems of characterization of convex bodies with sections and projections satisfying symmetry conditions ([MSR], [R1]). The problem can be reformulated in terms of functions on the sphere and various conditions on their restrictions to (large) sub-spheres.

As was partially discussed above, many curious questions deal with the visual characteristics of convex bodies. In particular, it would be extremely interesting to show analogous results for the questions involving **visual measures** instead of the shapes of supporting cones or their projections. The main difficulty in working with such questions is a lack of convenient notions (such as support function or radial functions) that work well for point projections. Nevertheless, different approaches are known. For instance, some involve techniques of integral geometry ([KO]).

Among problems regarding point projections, there are many interesting questions regarding the structure of **shadow boundaries** of convex bodies, [Bl1]. For example, it is known that ellipsoids are the only convex bodies in $\mathbb{E}^d$ that have flat shadow boundaries under orthogonal projections. It is very natural to ask whether the same holds true for point projections in any dimension. For example, it is possible to show that if a shadow boundary of a body is flat for point projections from sufficiently far away, then it is an ellipsoid indeed. Other relative positions of point projections for visual recognition of convex bodies with flat shadow boundary yield no known results.

Except the questions above, there are still a lot of motivating open problems about beautiful properties of convex bodies that hold true only for ellipsoids (check out Figure 7) For instance, K. Bezdek asks whether a convex body with *all* sections having an axis of symmetry is an ellipsoid or a body of revolution.

In my opinion one of the most elegant problems of convexity is the **Shephard problem** ([Sh]): if K and L are centrally symmetric convex bodies in $\mathbb{E}^d$ such that the volume of $K|H$ is smaller than the volume of $L|H$ for any subspace H of a fixed dimension, does it follow that the volume of K is smaller than that of L? It would be quite fascinating to investigate an analogous question for **spherical projections**. The case of equality for the volumes of the spherical projections from any point on the sphere (which corresponds to Aleksandrov theorem for orthogonal projections, see [Ga]) is of particular interest, as, possibly, a new approach to the questions of uniqueness.

It is worth noting here that particular cases of these and many other related questions can serve as fruitful undergraduate projects as well. This proved to be productive during Access and Support for Successful Undergraduate Research Experiences program in Summer 2017 at Kent State University, when my colleague and I worked on "baby versions" of convex geometry problems with the participants.

Since my Ph.D. studies, I have been interested in applications of harmonic analysis to the problems of convex geometry. I have been trying to attack an open question related to **Busemann-Petty problem** which is a "dual" counterpart of the Shephard problem. It asks whether for two centrally symmetric convex bodies K and L in $\mathbb{E}^d$, such that the volume of section $K \cap H$ is smaller than the volume of section $L \cap H$ for any subspace H of a fixed dimension, it follows that the volume of K is smaller than the volume of L (see [Ko])? The question about sections of dimension 2 and 3 remains open for $n \geq 5$.

There are many ways to generalize the result of Grünbaum ([Gr]) mentioned above. My collaborators and I have had some fruitful discussions for some particular cases. For instance, instead of considering the center of mass of a whole convex body, it could be interesting to ask the same question for the center of mass of its surface. Other **intrinsic volumes** are of an interest as well. A standard technique to tackle this problem deals with rearrangements (in particular, symmetrizations) that preserve the quantity of interest (volume, area etc.), and allow one to investigate the behaviour of centroids. This may be far from an easy task, especially for convex sets in non-Euclidean settings. A different way suggests that an appropriate embedding of sets in Euclidea space and investigating push-forward measures can be quite useful ([BHPS]).

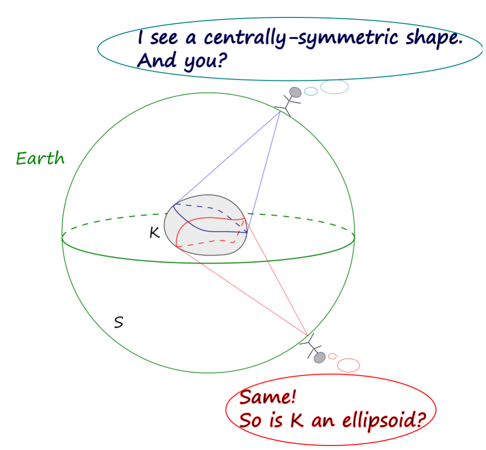


FIGURE 7. Is an ellipsoid the only convex body that looks centrally symmetric from any position?

Since the beginning of my postdoc, I have also been studying different aspects of probability and their relations to convex geometry and data science ([V]). In particular, one of my collaborators and I have worked on generalizing Khinchine's inequality ([MS]) for **dependent random variables**. At this point, except the fore-mentioned concentration inequalities and their statistical applications, we have partial results for particular types of dependencies.

It also seems very likely that the probability approach can be useful in the proof of the **inverse Brunn's theorem** for *non-convex bodies*, which has been one of my recent projects as well. It is known that ([Ba]), volume of cross-sections in a fixed direction of a d-dimensional convex body is a $\frac{1}{d-1}$-concave function on its support. According to numerical approximations, my colleague and I showed that the inverse of this statement does not hold. However, analytical justification remains unclear. For this reason, we consider a probabilistic approach: creating a probability space of the set of our potential counterexamples and investigating measures of different outcomes.

My last (and definitely not least) current ongoing project deals with interesting problems regarding **delocalization of random vectors**. My collaborators and I have been working on a problem related to *random hyperplanes* and *eigenvectors of random matrices* ([RV]). Namely, we are investigating the behaviour of the p^{th}-norm of a unit vector which is orthogonal to a random hyperplane spanned by vertices of a unit cube. Our main goal is to prove a concentration inequality for the norm. For this purpose, one of the powerful tools that we use is the mass transport ([Vi]) between two particular distributions on the unit sphere. Except being very curious pure math problems, questions of delocalization found fundamental applications in compressive sensing ([NV]).

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